About Fibonacci's Book of Squares

How elementary tools can solve quite elaborate problems

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Abstract

The main aim of the workshop was to read and discuss about results and proofs, which are to be found in Fibonacci's Book of Squares, (Liber quadratorum, Pisa, 1225), which work the author devoted to the solution he brought for Johannes of Palermo's question:

"Find a square number from which, when five is added or subtracted, always arises a square number"

Fibonacci offers material to his readers in a systematic way, orders things from the easiest to the more difficult and gives a proof for any result he appeals to.

It seems that, according to their school level, our pupils may be able to understand these results and proofs either through an inductive way of thinking or through a strict way of laying down the line of argument, if required.

For further ambition, Fibonacci's treatise provides material to reflect

- *on limits of natural language and the way complex calculations are carried out more easily with symbols,*
- *on the efficiency of elementary tools to solve quite elaborate problems especially arithmetical ones,*
- *on the way ancient texts can bring historical information about their author, time and topic, and above all throw light on unusual and consequently reputed difficult questions.*

1 Introduction

Leonardo of Pisa (1170–1240), known as Fibonacci, had an opportunity to learn the Indian art of calculation, as a teenager when staying in Algeria with his father, and as a young man while sailing along from one Arabic Mediterranean country to another for his own business trips. It seems he returned to Pisa when about 30. His most popular work is *Liber Abaci* (1202, 1228), which Fibonacci himself describes as:

> A book of fifteen chapters which comprises what I feel is the best of the Hindu, Arabic, and Greek methods, with proofs to further the understanding of the reader and the Italian people.

King of Sicily Frederic II (1194–1250), the grand son of red-bearded Frederic I, was raised Germanic emperor in 1212 and enjoyed being surrounded by a circle of fine scholars. He met Fibonacci at the time he held court in Pisa, about 1225. Scholar Johannes of Palermo took the opportunity to submit to Fibonacci the upper question:

> Find a square number from which, when five is added or subtracted, always arises a square number.

This question was in circulation at the time, but it is difficult to say whether Johannes of Palermo got it from the arithmetical tradition or from the algebraic one. We know that in the 10^{th} century al-Khazin and al-Karaji were involved in this question, but every one in his own way, respectively the arithmetical one and the algebraic one. Both got quite familiar with this kind of Diophantine problems after reading Diophantus' Arithmetica in Ibn Luqa's translation into Arabic about 900 but they had quite a different understanding of what Diophantus' work was.

Anyway the genuine and smart answer Fibonacci gives for this question in *Liber Quadratorum* (Pisa, 1225) is quite independent of any previous answer.

Sigler's English translation *The Book of Squares* is a set of twenty-four statements (although *Liber Quadratorum* does not present any separations) and Fibonacci actually solves two problems in it. The solution for the upper one comes out at proposition 17. Whereas the whole treatise culminates at proposition 24 in the solution Fibonacci brings for the difficult question proposed to him by Master Theodore, Philosopher to the Emperor:

> I wish to find three numbers, which added together with the square of the first number, make a square number. Moreover, this square, if added to the square of the second number, yields thence a square number. To this square, if the square of the third number is added, a square number similarly results.

It is not our purpose to study that second question within this article. Let us concentrate on the first one, which Fibonacci precisely introduces in the prologue for *The Book of Squares*:

> After being brought to Pisa by Master Dominick to the feet of your celestial majesty, most glorious prince, Lord Frederick, I met Master John of Palermo; he proposed to me a question that had occurred to him, pertaining not less to geometry than to arithmetic: find a square number from which, when five is added or subtracted, always arises a square number. Beyond this question, the solution of which I have already found, I saw, upon reflection, that this solution itself and many others have origin in the squares and the numbers which fall between the squares.

The announcement is very clear: the question is both arithmetical and geometrical. Fibonacci's solution is based on a very fresh consideration of the Euclidean properties and a very keen intuition of what we now call "the number theory". Fibonacci knows the property of squares as sums of odd numbers. He also knows the rules for the ordered sums of the squares of running from 1 consecutive or odd numbers. Whereas these results themselves are not new, their proofs are and all are to be seen on Euclidean line segment figures.

2 Two square numbers which sum to a square number

Let us start with the end:

We said the question gets solved at proposition 17. Fibonacci begins proposition 17 by writing:

> Here is the question mentioned in the prologue of this book.
>
> I wish to find a square number which increased or diminished by five yields a square number.

$$\text{(Modern writing: } c^2? \quad c^2 - 5 = x^2 \quad \& \quad c^2 + 5 = z^2)$$

and he goes on with technical advice leading to the solution.

But the main thing here is that he does understand the foundation of his solution and he is able to solve any similar problem. The complete question has been asked before at proposition 14:

> Find a number which added to a square number and subtracted from a square number yields always a square number.
>
> And thus must be found three squares and a number so that the number added to the smallest square makes the second square, and the same number added to the second square makes the third square, which is the greatest. And thus adding this number to, and subtracting it from, the second square yields always a square.

$$\text{(Modern writing: } N? \quad c^2 - N = x^2 \quad \& \quad c^2 + N = z^2$$
$$x^2, c^2, z^2, N? \quad x^2 + N = c^2 \quad \& \quad c^2 + N = z^2)$$

Let us restart with the beginning and read Fibonacci's introduction with the key in it:

> I thought about the origin of all square numbers and discovered that they arise out of the increasing sequence of odd numbers; for the unity is a square and from it is made the first square, namely 1; to this unity is added 3, making the second square, namely 4, with root 2; if to the sum is added the third odd number, namely 5, the third square is created, namely 9, with root 3; and thus sums of consecutive odd numbers and a sequence of squares always arise together in order.

$$\text{(Modern writing: } \sum_{1}^{n} (2k-1) = n^2)$$

There is no proof for this before proposition 4:

> I wish to demonstrate how a sequence of squares is produced from the ordered sums of odd numbers which run from 1 to infinity.

And the proof is not the one we expect. Since, according to proposition 2,

> [...] any square exceeds the square immediately before it by the sum of the roots of these squares.

$$\text{(Modern writing: } n^2 + [n + (n+1)] = (n+1)^2)$$

Fibonacci's proof consists in recognizing that the sequence of these sums is exactly the sequence of consecutive odd numbers.

From this introduction to proposition 3, Fibonacci gives a pack of results, all of them based on the upper key.

Proposition 1 contains several rules to find two square numbers which sum to a square number. Fibonacci explains all of them in full text with help of numerical examples. But rules and examples are general and therefore consistent with symbolic writing. It should be taken for granted that modern writing we choose to use in this article is anachronistic but suitable with Fibonacci's theories.

> Hence, to find two square numbers which sum to a square number, I shall take any odd square and I shall have it for one of the two said squares; the other I shall find in a sum of all odd numbers from unity up to the odd square itself. For example, I shall take 9 for one of the mentioned two squares, ...

$$\begin{aligned}&(2p-1)^2 = \left[2\left(2p^2-2p+1\right)-1\right]\\&1+3+\ldots+\left[2\left(2p^2-2p\right)-1\right]+\left[2\left(2p^2-2p+1\right)-1\right]\\&\left[2p^2-2p\right]^2+(2p-1)^2=\left[2p^2-2p+1\right]^2\end{aligned}$$

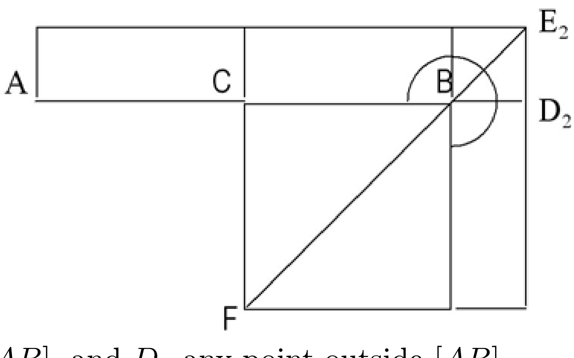
Fibonacci goes on studying different possibilities for the square added to be the sum of two, three, four, ... consecutive odd numbers.

Next result is very important for the solution of John of Palermo's question. The proof for it is to be read on a linear Euclidean figure. Here are text and figure one under the other:

> Similarly, it is demonstrated that any square exceeds any smaller square by the product of the difference of the roots by the sum of the roots. For example, let *.ag.* and *.gb.* be two roots of any two squares whatsoever, and let *.gb.* be bigger than *.ag.* by the number *.db.* Because the product of *.ag.* with itself, plus the product of *.db.* with *.ab.,* equals the product of *.gb.* with itself, the square of *.gb.* exceeds the square of *.ag.* by as much as the root *.gb.* exceeds the root *.ag.* multiplied by the sum of *.gb.* and *.ag.,* namely, by the product of *.db.* and *.ab.* This is what had to be demonstrated.

Nearly modern writing: $.ag.^2 + .ba. \times .bd. = .gb.^2$

So the proof here consists in recognising that *.ba.* is the sum and *.bd.* is the difference of the roots. The base implicitly referred to is Euclid, Book II, proposition 6. Let us recall what it says on this figure, which looks like those generally ascribed to Euclid:

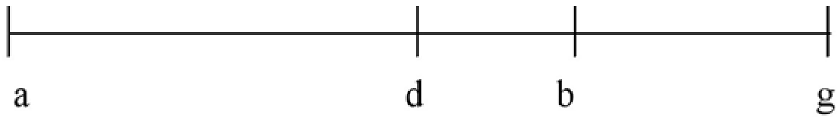


Let C be the middle of $[AB]$, and D_2 any point outside $[AB]$
Square FB + Gnomon = Square FE_2
Since Gnomon = Rectangle AE_2, Square FB + Rectangle AE_2 = Square FE_2
What can be written in a more modern geometrical way $CB^2 + D_2A \times D_2B = CD_2^2$
(that is the result Fibonacci appeals to)
and for a quite modern algebraic extension
with $D_2A = a$ and $D_2B = b$, $\left(\frac{a-b}{2}\right)^2 + ab = \left(\frac{a-b}{2} + b\right)^2$ or $\left(\frac{a-b}{2}\right)^2 + ab = \left(\frac{a+b}{2}\right)^2$

That was Fibonacci's geometrical proof for what we got used to consider as a nearly obvious algebraic formula, whereas it is a very essential point in Fibonacci's solution

$$x^2 + (y-x)(y+x) = y^2$$

One more rule at proposition 3: Fibonacci gives "another way of finding two squares which make a square number with their sum". The geometrical argument obviously refers to Euclid, Book II, proposition 5, in case of $.ba.$, $.bg.$ being squares. $.ag.$ is divided in two equal parts by .d..



$$.ba. \times .bg. + .db.^2 = .dg.^2$$

which, in modern algebraic language, with $.ba. = a^2$ and $.bg. = b^2$, is not different from

$$a^2b^2 + \left(\frac{b^2+a^2}{2} - a^2\right)^2 = \left(\frac{b^2+a^2}{2}\right)^2$$

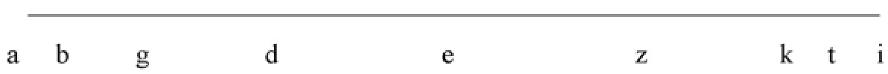
Propositions 5 to 9 are devoted to results about equalities between sums of squares, the sums themselves being either squares or not. These are not directly useful for our article topic.

3 Multiples of 24 and congruous numbers

Proposition 10 is a very interesting one, both for the result coming out and for the kind of proof it does use. Is this proof a real mathematical induction or not? Should we teach our pupils with intuitive methods? Are very strict formulations always necessary, according to tests of exactness?

> If, beginning with the unity, a number of consecutive numbers, both even and odd numbers, are taken in order, then the triple product of the last number and the number following it and the sum of the two, is equal to six times the sum of the squares of all the numbers, namely from the unity to the last.

$$\text{Modern writing: } k\,(k+1)\,[k+(k+1)] = 6\sum_{1}^{k} i^2$$



$$.ab. = 1 \quad .bg. = .ab. + 1 \quad .zi. = .ez. + 1 \quad .ez. = .zt. \quad .ti. = 1$$
$$.zt. = .ez. = .de. + 1 \quad .de. = .zk. \quad .kt. = 1 \quad .ki. = 2$$

The proof is in two parts; the first one to establish how one triple product is linked to the one just before. Here is the link which gets proved at the end of the first part

$$.ez..zi..ei. = .de..ez..dz. + 6.ez.^2$$

$.ez..zk..ek. = .de..ez..dz.$
$.ez..zk..ek. + .ez..zk..ki. + .ez..ki..ei. =$
$.ez..zk..ei. \; + .ez..ki..ei. =$
$.ez..zi..ei.$
$.ez..zk..ek. + .ez..zk..ki. + .ez..ki..ei. =$
$.ez..zk..ek. + 2.ez.(.ez. - 1) + 2.ez.(2.ez. + 1) =$
$.ez..zk..ek. + 2.ez.^2 - 2.ez. + 4.ez.^2 + 2.ez. =$
$.ez..zk..ek. + 6.ez.^2$
$.ez..zi..ei. = .de..ez..dz. + 6.ez.^2$

Here is the demonstration for the link. Be careful about the fact that the opposite writing, which looks algebraic, is not. I'd like it to be the exact transcription of Fibonacci's full sentences. Calculation involves segments and the equalities are to be read on the upper figure.

The second part of the proof consists in going down step by step from the last number to the first (unity) and so gathering the expected result one piece after the other.

$$
\begin{aligned}
&.ez..zi..ei. = .de..ez..dz. + 6.ez.^2\\
&.de..ez..dz. = .gd..de..ge. + 6.de.^2\\
&.ez..zi..ei. = .gd..de..ge. + 6\left(.de.^2 + .ez.^2\right)\\
&.gd..de..ge. = .bg..gd..bd. + 6.gd.^2\\
&.ez..zi..ei. = .bg..gd..bd. + 6\left(.gd.^2 + .de.^2 + .ez.^2\right)\\
&.bg..gd..bd. = .ab..bg..ag. + 6.bg.^2\\
&.ez..zi..ei. = .ab..bg..ag. + 6\left(.bg.^2 + .gd.^2 + .de.^2 + .ez.^2\right)\\
&.ab..bg..ag. = 1 \times 2 \times 3 = 6 = 6.ab.^2\\
&.ez..zi..ei. = 6\left(.ab.^2 + .bg.^2 + .gd.^2 + .de.^2 + .ez.^2\right)
\end{aligned}
$$

In proposition 11, Fibonacci presents a few extensions of this result, proving that he is able to catch the largest and deepest meaning of what is asked and what he does. These will be helpful to find good multiples of 24 to be congruous numbers.

> If, beginning with the unity, a number of consecutive odd numbers are taken in order, then the triple product of the last number and the odd number following it and their sum is equal to twelve times the sum of all the squares of the odd numbers from the unity to the last odd number [...]
>
> By a similar method, if beginning with the number two, consecutive even numbers are taken in order, the triple product of the last of them, the number following it, and the sum of the two [...]
>
> By the same way and method again, if consecutive multiples of three are taken in ascending order beginning with three, [...]

$$
\begin{aligned}
(2k-1)(2k+1)\left[(2k-1)+(2k-1)\right] &= 2 \times 6 \sum_{1}^{k} (2i-1)^2\\
kr\,(kr+r)\left[kr + (kr+r)\right] &= r \times 6 \sum_{1}^{k} (ir)^2
\end{aligned}
$$

Proposition 12 is an approach of what congruous numbers can be. Specific quadruple products of two numbers, their sum and their difference are multiples of 24, and Fibonacci declares them congruous numbers.

> If two numbers are relatively prime and have an even sum, and if the triple product of the two numbers and their sum is multiplied by the number by which the greater number exceeds the smaller number, there results a number which will be a multiple of twenty-four [...]
>
> And if one of the numbers *.ab.* and *.bg.* is even, the sum of them will be odd; then it will be similarly shown that from the product of the doubles of each of the numbers and their sum and the number *.dg.* will arise a number which will be a multiple of twenty-four whether the numbers are relatively prime or not. This obtained number, namely the multiple of twenty-four, is called congruous.

And the way to prove it is easy, as far as things are well ordered. We'll write the proofs in a modern way, in full respect of Fibonacci's text.

Let a, b be the two numbers such that $\gcd(a,b) = 1$ $a < b$

I. $(b+a) \cong 0\,[2]$
$a \cong 1\,[2]\,\&b \cong 1\,[2]\,\&\,(b+a-2a) =$
$(b-a) \cong 0\,[2]$
1. $\frac{1}{2}(b-a) \cong 1\,[2]$
$\frac{1}{2}(b-a) \cong 1\,[2]\,\&a \cong 1\,[2] \Rightarrow \frac{1}{2}(b+a) \cong 0\,[2]$
$(b+a) \cong 0\,[4]\,\&\,(b-a) \cong 0\,[2] \Rightarrow$
$(b+a)(b-a) \cong 0\,[8]$
2. $\frac{1}{2}(b-a) \cong 0\,[2]$
$(b-a) \cong 0\,[4]\,\&\,(b+a) \cong 0\,[2] \Rightarrow (b+a)(b-a) \cong 0\,[8]$

$*a \cong 0\,[3]$ or $b \cong 0\,[3]$
$*a \cong 1\,[3]\,\&\,b \cong 1\,[3]$ or $a \cong 2\,[3]\,\&\,b \cong 2\,[3]$
$(b-a) \cong 0\,[3]$
$*a \cong 1\,[3]\,\&b \cong 2\,[3]$
$(b+a) \cong 0\,[3]$
In all three cases, $ab(b+a)(b-a) \cong 0\,[3]$

II. $(a+b) \cong 1\,[2]$
$a \cong 1\,[2]\,\&\,b \cong 0\,[2]\,\&\,(a+b) \cong 1\,[2]\,\&\,(b-a) \cong 1\,[2]$
$ab(b+a)(b-a) \cong 0\,[2]$
$2a2b(b+a)(b-a) \cong 0\,[8]$

As a conclusion: If $(b+a)$ is even, $ab(b+a)(b-a) \cong 0$ [24]. It is a congruous number
If $(b+a)$ is odd, $2a2b(b+a)(b-a) \cong 0$ [24]. It is a congruous number

4 Stairs of consecutive odd numbers

Proposition 13 presents an elementary result, able to do great things. An illustration will be enough for a proof. I must say the stairs are mine, useful for the transcription of Fibonacci's text.

> If about some given number are located some smaller and larger numbers and if the number of smaller numbers equals the number of larger numbers, and if each of the larger numbers exceeds the given number by the same as the given number exceeds a smaller number, then the sum of all the smaller and larger numbers will be the product of the number of located numbers and the given number.

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & A-r_1 & & A+r_1 & & & \\ & & A-r_2 & & & & A+r_2 & & \\ & A-r_k & & & & & & A+r_k & \end{array}$$

$$(A-r_k)+\ldots+(A-r_2)+(A-r_1)+(A+r_1)+(A+r_2)+\ldots+(A+r_k) = 2kA$$

For example:

$$\begin{array}{ccccccccccc} & & & & & 2k & & & & & \\ & & & & 2k-1 & & 2k+1 & & & & \\ & & & 2k-3 & & & & 2k+3 & & & \\ & & \text{É} & & & & & & \text{É} & & \\ & [2k-(2p-1)] & & & & & & & & 2k+(2p-1)] & \\ 2k-2p & & & & & & & & & & 2k+2p \end{array}$$

$$[2k-(2p-1)]+\ldots+(2k-3)+(2k-1)+$$
$$(2k+1)+(2k+3)+\ldots+[2k+(2p-1)] = 2p \times 2k$$

which is the sum of the $(2p)$ consecutive odd numbers comprised between the two even numbers $(2k-2p)$ and $(2k+2p)$.

It seems we are ready for proposition 14, the one that asks the general question to be solved, as we already noticed

> Find a number which added to a square number and subtracted from a square number yields always a square number.
>
> And thus must be found three squares and a number so that the number added to the smallest square makes the second square, and the same number added to the second square makes the third square, which is the greatest. And thus adding this number to, and subtracting it from, the second square yields always a square.

$$x^2,\ c^2,\ z^2,\ N?\ x^2 + N = c^2 \& c^2 + N = z^2$$

Sigler's name for$N$ will be a congruous number and for c^2 it will be a congruent square.

Fibonacci here starts a long and not so easy demonstration in four parts, one for every possible case. He first gives general explanations and appeals to numerical examples only

> to see these things still more clearly

I'll give a general presentation for the only first and main part, and then apply the rule for every numerical example.

$$\sum_1^x (2i-1) = x^2 \quad \sum_1^c (2i-1) = c^2 \quad \sum_1^z (2i-1) = z^2$$
$$(2\times 1 - 1) + (2\times 2 - 1) + \ldots + [2(x-1) - 1] + (2x-1) = x^2$$

is the sum of x consecutive odd numbers beginning with the unity

$$[2(x+1) - 1] + \ldots + [2(c-1) - 1] + (2c-1) = c^2 - x^2$$

is the sum of the $(c-x)$ following consecutive odd numbers comprised between the two even $2x$ and $2c$.

$$[2(c+1) - 1] + \ldots + [2(z-1) - 1] + (2z-1) = z^2 - c^2$$

is the sum of the $(z-c)$ following consecutive odd numbers comprised between the two even $2c$ and $2z$.

It is wished that the sum of the $(c-x)$ middle ones equals the sum of the $(z-c)$ last ones.

> Let a and b, $a < b$, be two arbitrary numbers
> First suppose $(b+a)$ is even, which makes $(b-a) = [(b+a) - 2a]$ even too.
> Either $\frac{b}{a} < \frac{b+a}{b-a}$ or $\frac{b}{a} > \frac{b+a}{b-a}$. First suppose $\frac{b}{a} < \frac{b+a}{b-a}$ (1)
>
> $$\frac{b}{a} = \frac{b(b-a)}{a(b-a)} = \frac{b(b+a)}{a(b+a)}$$
>
> $[b(b-a)]\,[a(b+a)] = [a(b-a)]\,[b(b+a)]$ in which all factors are even.
> Let us set $[b(b-a)] = e$, $[a(b+a)] = f$, $[a(b-a)] = g$, $[b(b+a)] = h$
> It comes $e \times f = g \times h$ with $e < f$ according to (1)
> And it appears that $[b(b+a)] - [a(b+a)] = b^2 - a^2 = [b(b-a)] + [a(b-a)]$ that means $h - f = e + g \Leftrightarrow f + e = h - g$
> $e \times f$ is the value of the sum of the e consecutive odd numbers comprised between the two even numbers $f - e$ and $f + e$.
> $g \times h$ is the value of the sum of the g consecutive odd numbers comprised between the two even numbers $h - g$ and $h + g$.
> And it works because $f + e = h - g$. No odd number forgotten, none used twice.

The smallest square number looked for is $\left(\frac{f-e}{2}\right)^2 = \left[\frac{2ab-\left(b^2-a^2\right)}{2}\right]^2$ which is the sum of all consecutive odd numbers from the unity up to $(f-e-1)$. The middle square number looked for is $\left(\frac{f+e}{2}\right)^2 = \left(\frac{h-g}{2}\right)^2 = \left(\frac{b^2+a^2}{2}\right)^2$ which is the sum of all consecutive odd numbers from the unity up to $(f+e-1) = (h-g-1)$. This is the one which is called a congruent square.

The largest square number looked for is $\left(\frac{h+g}{2}\right)^2 = \left[\frac{2ab+\left(b^2-a^2\right)}{2}\right]^2$ which is the sum of all consecutive odd numbers from the unity up to $(h+g-1)$. The congruous number looked for is $e \times f = g \times h = ab\,(b+a)\,(b-a)$. It is the common value of both sums of intermediate consecutive odd numbers, whose quantities e and g have the same ratio one to the other as b has to a: $\frac{e}{g} = \frac{b}{a}$.

We'll now study Fibonacci's numerical examples and go into detail for each of them. If $(b+a)$ is even, and $\frac{b}{a} < \frac{b+a}{b-a}$

$$a = 3 \quad b=5 \quad b+a=8 \quad b-a=2 \quad \frac{5}{3} < \frac{8}{2}$$

$$\frac{5}{3} = \frac{5\times 2}{3\times 2} = \frac{5\times 8}{3\times 8} \qquad (5\times 2)\,(3\times 8) = (3\times 2)\,(5\times 8)$$

24
23 25
21 27 **40**
19 29 39 41
17 31 37 43
15 33 35 45
14 **34** **46**
1 . . . 13

$$7^2 + 10\times 24 = 17^2 \qquad 17^2 + 6\times 40 = 23^2$$

If $(b+a)$ is even, and $\frac{b}{a} > \frac{b+a}{b-a}$

$$a=1 \quad b=3 \quad b+a=4 \quad b-a=2 \quad \frac{4}{2} < \frac{3}{1}$$

$$\frac{4}{2} = \frac{4\times 1}{2\times 1} = \frac{4\times 3}{2\times 3} \qquad (4\times 1)\,(2\times 3) = (2\times 1)\,(4\times 3)$$

6
5 7 **12**
3 9 11 13
2 **10** **14**
1

$$1^2 + 4\times 6 = 5^2 \qquad 5^2 + 2\times 12 = 7^2$$

If $(b+a)$ is odd, and $\dfrac{b}{a} < \dfrac{b+a}{b-a}$,

$$a=1 \quad b=2 \quad b+a=3 \quad b-a=1 \quad \frac{2}{1} < \frac{3}{1} \text{ or } \frac{2\times 2}{1\times 2} < \frac{3}{1}$$

$$\frac{2\times 2}{1\times 2} = \frac{4}{2} = \frac{4\times 1}{2\times 1} = \frac{4\times 3}{2\times 3} \qquad (4\times 1)(2\times 3) = (2\times 1)(4\times 3)$$

Same products, same stairs and same equality $1^2 + 4\times 6 = 5^2 5^2 + 2\times 12 = 7^2$

Since 24 is coming out as a congruous number both from the smallest pair (a,b), $(a+b)$ even, and from the smallest pair (a,b), $(a+b)$ odd, it is the smallest congruous number which can be found this way.

If $(b+a)$ is odd, and $\dfrac{b}{a} > \dfrac{b+a}{b-a}$

$$a=2 \quad b=5 \quad b+a=7 \quad b-a=3 \quad \frac{5}{2} > \frac{7}{3} \text{ or } \frac{7}{3} < \frac{10}{4}$$

$$\frac{7}{3} = \frac{7\times 4}{3\times 4} = \frac{7\times 10}{3\times 10} \qquad (7\times 4)(3\times 10) = (3\times 4)(7\times 10) = 840$$

30
29 31 **70**
27 33 69 71
... ...
3 57 59 81
2 **58** **82**
1

$1^2 + 840 = 29^2$ $\qquad$ $29^2 + 840 = 41^2$

5 Solution as a conclusion

So, what do we know now?

Every $ab\,(b+a)\,(b-a)$, with the sum $(b+a)$ even, is a multiple of 24 and is a congruous number for the congruent square $\left(\dfrac{b^2+a^2}{2}\right)^2$.

Every $2a2b\,(b+a)\,(b-a)$, with the sum $(b+a)$ odd, is a multiple of 24 and it can be shown that it is a congruous number for the congruent square $\left(b^2+a^2\right)^2$.

And Fibonacci writes:

> The first congruous number that can be found with integral squares is 24, and from 24 are generated all congruous numbers.

Although al-Khazin did it before him, Fibonacci does not actually prove that all congruous numbers proceed from a pair (a,b) as shown before, and consequently are multiples of 24. But he goes on, producing "good multiples" of 24

> Indeed, how many times 24 shall be multiplied by a square number, as many congruous numbers will be produced.

$$1^2 + 24 = 5^2 \qquad \text{and} \qquad 5^2 + 24 = 7^2$$
$$(1k)^2 + 24k^2 = (5k)^2 \qquad \text{and} \qquad (5k)^2 + 24k^2 = (7k)^2$$
$$(5k)^2 - (1k)^2 = 4k \times 6k \qquad (7k)^2 - (5k)^2 = 2k \times 12k$$

where $4k$ and $2k$ are the respective quantities of odd numbers in each sequence of odd numbers which sum to the congruous number.

Next result allows us to find as many congruous numbers as wished.

> Similarly, a congruous number will result if 24 will be multiplied by a sum of squares which will be made of a sum of increasing numbers, both odd and even beginning with the unity, or by odd numbers only, or. . .

$$\begin{aligned}
24\sum_{1}^{k} i^2 &= 24\frac{k(k+1)(2k+1)}{6} = (2k)[2(k+1)](2k+1)\times 1 \\
24\sum_{1}^{k} (2i-1)^2 &= 24\frac{(2k-1)(2k+1)(4k)}{2\times 6} = (2k-1)(2k+1)(4k)\times 2 \\
24\sum_{1}^{k} (ir)^2 &= 24\frac{kr(kr+r)(2kr+r)}{r\times 6} = r^2(2k)[2(k+1)](2k+1)\times 1
\end{aligned}$$

At first sight proposition 15 does not bring much in regard of what has been done before, but it actually is one more step to the solution:

> If some congruous number and its congruent squares are multiplied by another square, the number made by the product of the congruous number and the square will be a congruous number. . .

$$\begin{gathered}
x^2 + N = c^2 \quad c^2 + N = z^2 \\
(xk)^2 + Nk^2 = (ck)^2 \quad (ck)^2 + Nk^2 = (zk)^2
\end{gathered}$$

Proposition 16 is the last step to the solution for John of Palermo's specific problem:

> I wish to find a congruous number which is a square multiple of five.

Let b be 5 and a be 2^2, so that $(b+a)$ and $(b-a)$ are squares. $b+a = 5+2^2 = 3^2$, which is odd $b-a = 5-2^2 = 1^2$ The congruous number produced by these a and b will be a multiple of five and a square

$$(2\times 2^2)\times(2\times 5)\times\left(5+2^2\right)\times\left(5-2^2\right) = 12^2\times 5 = 720$$

And at last, happy end at proposition 17.

> I wish to find a square number which increased or diminished by five yields a square number.

$$(2\times 2^2)\times(2\times 5)\times\left(5+2^2\right)\times\left(5-2^2\right) = 720 = 12^2\times 5$$

$$\frac{5}{4} < \frac{9}{1} \quad \frac{(2\times 5)\times 1}{(2\times 4)\times 1} = \frac{(2\times 5)\times 9}{(2\times 4)\times 9} \quad N = 10\times 72 = 8\times 90$$

$$c^2 = \left(\frac{72+10}{2}\right)^2 = \left(\frac{90-8}{2}\right)^2 = 41^2 \quad x^2 = \left(\frac{72-10}{2}\right)^2 = 31^2 \quad z^2 = \left(\frac{90+8}{2}\right)^2 = 49^2$$

$$31^2 + 12^2\times 5 = 41^2 \quad 41^2 + 12^2\times 5 = 49^2$$

$$\left(\frac{31}{12}\right)^2 + 5 = \left(\frac{41}{12}\right)^2 \qquad \left(\frac{41}{12}\right)^2 + 5 = \left(\frac{49}{12}\right)^2$$

$$\left(2\frac{7}{12}\right)^2 + 5 = \left(3\frac{5}{12}\right)^2 \qquad \left(3\frac{5}{12}\right)^2 + 5 = \left(4\frac{1}{12}\right)^2$$

John of Palermo's question is solved and we now know that 5 is a congruous number. We nearly knew that 6 is one and it can be shown that 7 is one too (with $b = 4^2$ and $a = 3^2$).

$$1^2 + 2^2 \times 6 = 5^2 \quad \text{and} \quad 5^2 + 2^2 \times 6 = 7^2$$

$$\left(\frac{1}{2}\right)^2 + 6 = \left(\frac{5}{2}\right)^2 \quad \text{and} \quad \left(\frac{5}{2}\right)^2 + 6 = \left(\frac{7}{2}\right)^2$$

It had nowhere been specified if we were looking for integers or rational numbers as a solution. But it is now confirmed that the main question is the one Fibonacci asked at proposition 14, looking for integers. Thanks to his clear and well-ordered treatise, we are able to "make" congruent pairs of integers and tabulate lists of them.

But even nowadays nobody can prove that the numbers conjectured as the congruous numbers actually are the congruous numbers.

So let us set apart this not exhausted theoretical question, and enjoy the matter available in Fibonacci's treatise, many results and various proofs, for us and for our pupils, at any chosen level.

References

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