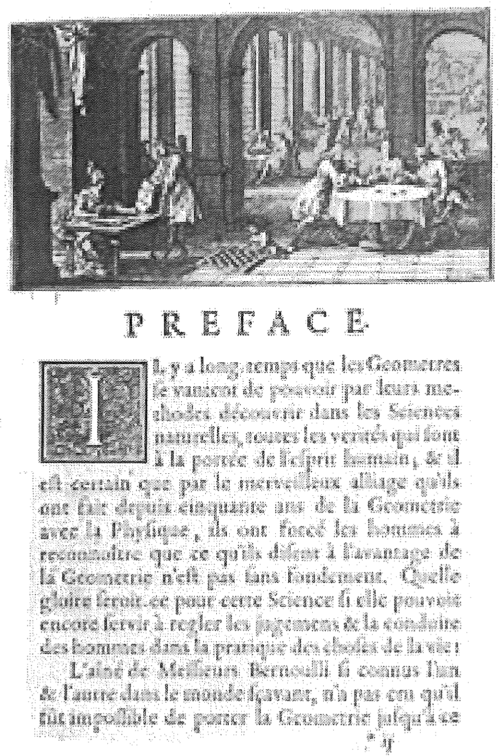

PREFACE.

IL y a long-temps que les Geometres se vantent de pouvoir par leurs methodes découvrir dans les Sciences naturelles, toutes les veritez qui sont à la portée de l'esprit humain; & il est certain que par le merveilleux alliage qu'ils ont fait depuis cinquante ans de la Geometrie avec la Physique, ils ont forcé les hommes à reconnoître que ce qu'ils disent à l'avantage de la Geometrie n'est pas sans fondement. Quelle gloire seroit-ce pour cette Science si elle pouvoit encore servir à regler les jugemens & la conduite des hommes dans la pratique des choses de la vie!

L'aîné de Messieurs Bernoulli si connus l'un & l'autre dans le monde sçavant, n'a pas cru qu'il fût impossible de porter la Geometrie jusqu'à ce

Première page de l'*essai d'Analyse sur les jeux de Hazard* (1713) de Montmort

On the duality of the probability concept – from the epistemological point of view

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Abstract

The dual character of probability concept -described from historical point of view by Ian Hacking- has become an inspiration to me for research in epistemology of probability in the process of learning in today's classroom.

In this presentation I would like to show and to analyse students' natural ways of probabilistic thinking and forming probabilistic mental objects which can be matematised by the mature dual concept of probability. The essential feature of this concept -duality- serves as a tool for diagnosis and evaluation of a degree of maturity of student's probabilistic knowledge and understanding. In order to recognise the main features of the process of probability learning many didactical situations in various cases will be carefully considered. Participants will have an opportunity to experience these situations.

1 Introduction

The dual character of the probability concept –described from the historical point of view by Ian HACKING (1975)– has become an inspiration to me for research in epistemology of probability in the process of learning in today's classroom. In this article I would like to show and to analyse students' natural ways of probabilistic thinking and forming probabilistic mental objects which can be mathematised by the mature dual concept of probability. The essential feature of this concept –duality– serves as a tool for diagnosis and evaluation of the degree of maturity of the student's probabilistic knowledge and understanding. In order to recognise the main features of the process of probability learning many didactical situations in various cases are carefully considered.

2 The duality of the probability concept from the historical viewpoint

Ian HACKING (1975) –analysing old authentic or reconstructed probabilistic reasonings– argues that *the concept of probability* in its historical development *has a dual nature*. He distinguishes two aspects of this concept. One of them *–epistemological–* is implied by the general state of our knowledge concerning a phenomenon under consideration, and is related to the degree of our belief, conviction or confidence, in an argument. The other aspect *–aleatory–* is related to the physical structure of the random mechanism and to its tendency to produce stable relative frequencies of events. The first aspect gives the basis for the "chance calculus" and the other for the "frequency calculus". The analysis of old probabilistic reasonings shows that both these aspects became inseparable starting from the time of Pascal (about 1660). Before that time they were developed independently. Consciously joining them together, which was made mainly thanks to Blaise Pascal and Pierre Fermat – has illuminatied probabilistic thinking. From that moment the concept of probability acquired a dual, mathematically mature, character. (LAKOMA 1992, 1999a)

However, when we analyse old probabilistic reasonings from the pre-Pascal time, we can recognise in some of them ways of thinking and mental objects which are predecessors of the dual probability. The most spectacular examples, which can be treated as anticipating further evolution of probability and indicating the right directions leading to a crystallisation of dual probability, we can meet in works of Galileo or Cardano (HACKING 1975, LAKOMA 1992). It is important to notice that both authors –using in their reasonings those mental objects which referred to different aspects of probability– tried *to gain a balance* in two-sided arguments. Feeling of such balance brought them sufficient *explanatory value* of their arguments and let them find the right solution.

For example, let us observe Galileo's solution of a problem of throwing three dice and taking into account sums of points (HACKING 1975). Galileo reports that somebody noticed inconsistency between two opinions concerning that game : "With three dice 9 and 12 can be made up in as many ways as 10 and 11. Each can be decomposed into 6 partitions. However *it is known from long observation* that dice players consider 10 and 11 to be more advantageous than 9 and 12".

Galileo decided that "there is a very simple explanation, namely that *some numbers are more easily and more frequently made than others*, which depends on their *being able to be made up with more variety of numbers*". In particular, 6 ways of obtaining sum of 9 and 12 can be acquired by 25 permutations of results of throwing three dice. Similarly, 6 ways of obtaining

10 and 11 can be acquired from 27 permutations. If all these permutations *are equally easy to obtain*, then 11 is more advantageous than 12 in the ratio 27:25.

Galileo posed the hypothesis that permutations are equally probable, against the hypothesis that ways of obtaining sums (partitions) are equally probable. The second hypothesis is inconsistent with the facts, whereas the first one fits the facts exactly. This way of reasoning we can consider as an example of refuting statistical hypothesis on the base of long observation. This way of thinking also shows its general idea – to make some observations of random phenomena and to confront them with a model. If the model does not fit to the reality, there is a need to build another model, which will better fit the phenomenon under consideration. Observing experimental frequencies of outcomes is confronted with estimating *chances* of obtaining these outcomes. These chances or *theoretical frequencies* are estimated by means of the *propensity* or *facility* of random mechanisms to produce them.

Also Cardano –in his solutions of problems concerning throwing two dice (HACKING 1975, LAKOMA 1992), which were presented in his work "De ludo aleae" (1550)– makes *idealisation* of a problem and searches for *symmetry* of dice : "I am able to throw 1, 3 or 5 *equally easy* as 2, 4 or 6. Therefore it is possible to make some bets according to this equality, if dice are honest". Cardano is convinced that probability is connected with a *propensity (tendency)* of a die to produce stable frequencies in repeating trials. Galileo's opinion on some results that are easier and more frequently appear also refers to this *proclivity* of a die.

Thus, in the solutions quoted above, we can notice that there is a natural need to adduce arguments based on physical features of random mechanisms and to support them by arguments related to a state of knowledge or conviction concerning this mechanism. Both these aspects, epistemological and aleatory, are present clearly enough; also a tendency to support arguments based on one aspect by arguments based on the other seems to be strong. Finding a balance between them leads to formulating conclusions and to the solution of a problem. It is also worth remembering, that the word "probability" was not used in pre-Pascal time to solve problems as described above, but all mental objects like propensity, facility, tendency, chances or theoretical frequencies belonged to pre-origins and pre-conditions of the dual probability.

History shows that in order to acquire the probability concept it is necessary to make conscious its dual nature. Therefore I believe that in the process of probability teaching –from the very beginning of education– it is necessary to create such conditions that it will be possible to form in the student's mind the dual probability concept (LAKOMA 1990).

In my educational research this fundamental feature of probability has become a basis for investigating the process of developing students' probabilistic concepts and reasonings. I search for some symptoms of understanding in student's probabilistic arguments, and also for mental objects, which the student creates at various stages of his cognitive development and by means of which he forms in his mind the dual probability concept in a proper way.

In order to achieve this aim, it is necessary to create such educational conditions that the process of probability learning could be developed according to student's cognitive development.

3 Dual concepts probability in the process of stochastics learning

Careful observation of a process of mathematics learning shows that the process of probability learning can be naturally developed when there is a need to solve a problem (LAKOMA 1990, 1998). When analysing students' work, it is possible to distinguish four main steps of a process of problem solving :

> discovering and formulating a problem; constructing a model of the "real" phenomenon; analysing of a model; confronting results obtained from a model with the "real" situation.

This way arises directly from student's common sense thinking while considering simple problems as well as more advanced ones. When we look at history, we find out that this methodology comes from Isaac Newton (1678).

Mathematical models used by students are usually as simple as possible, have a local character and a strong *explanatory value*. These are *the local models*. However, what is essential is the general way of reasoning and acting which can be developed in a unique way on every level of mathematics learning. Only the classes of problems considered, and the mathematical tools useful for solving, can change on various levels of education (LAKOMA 1990).

The main aim of probabilistic education is to create such didactical situations that students will have an opportunity to solve probabilistic problems by means of those mathematical tools which are at hand for them. In the further part of this article I will present some examples of such mathematical activities in which we follow students' probabilistic reasonings and use of mental objects. In particular, we will observe in what way the duality of the probability concept is formed in the student's mind.

During analysis of students' work it would be profitable to distinguish two aspects of probability, using the methodology of Leibniz (HACKING 1975). Leibniz identifies epistemological aspect of probability with arguments *de dicto*, which means – with arguments referred to this, what we know about the phenomenon and what we would express. On the other hand, an aleatory aspect is identified by arguments *de re*, that means by arguments concerning physical characteristics of a random phenomenon. Thus, mental objects such as *experimental frequencies* express an *aleatory* side of probability, whereas *facilities, theoretical frequencies, chances* –based on analysis of a model– express the *epistemological* side of probability. However, we must remember that carefully supporting each of these sides are symptoms of understanding the *dual* probability concept.

For an example of how we can recognise different aspects of probability in students' reasonings, let us consider the simplest random phenomenon – the throw of a coin. Why do we argue that the probability of throwing a head is $\frac{1}{2}$? When we pose this question to our students, we usually obtain various arguments like the following : 1) *Probability is $\frac{1}{2}$ because there are two possible outcomes.* 2) *A coin has two sides – both are equal (the same), so the probability of obtaining a head is one half.* 3) *When we throw a coin many, many times, we will obtain one half of heads.* There are also some doubts : 4) *There is also another outcome – when a coin stands up on its edge.*

Students often give all these arguments. It is easy to notice that argument 1) is not sufficient to answer the question correctly. In this case it is necessary to add that we consider a coin as symmetric one. Only this conviction justifies the probability given in the question. But it is very symptomatic that argument 2) is also not sufficient for many of students. They feel a need to find a support for this *epistemological* consideration – they analyse the throw of a coin from the *aleatory* point of view, by observing experimental frequencies or by trying to predict their values in a long observation of this random mechanism. When students use both arguments 2) and 3) they usually are satisfied with their reasonings and they decide to formulate final conclusions. If argument 4) appears in student's reasoning, it seems to be a symptom of *lack of*

balance in both sides of arguments. We can observe an excessive attachment to considerations *de re* – concerning physical structure of a random mechanism. This disturbs correct thinking.

This example also shows the natural way of presenting and exploring a random mechanism – students in their arguments usually describe it by specification of possible outcomes and predicting chances of their appearing.

When we –mathematics teachers– use the mature probability we certainly do not distinguish and do not expose explicitly two aspects of this concept in our concept reasonings. Our students do not have the dual concept of probability at their disposal yet. Thus we are able easily to recognise in their reasonings both probability aspects, which appear separately and independently of each other. The way in which students use these aspects shows us what is a degree of advance of their probabilistic thinking and what is their stage of forming dual probability. Reaching a balance in using both arguments *de re* and *de dicto* usually shows us that their probability concept is developing properly and that they are able to gain a relational understanding of probabilistic concepts such as probability, or frequency.

Let us consider some examples of probabilistic activities of students.

4 Examples of didactical situations

A. "Once upon a time. . . "

The picture (Figure 1) presents the Market Square in Cracov (the former capital city of Poland) on the 15^{th} of July 1410. Everybody in Poland knows this date. This is the date of the battle of Grunwald (far from Cracov) in which Polish and Lithuanian forces –commanded by the Polish king Vladislav Jagiello– fought against the Teutonic Knights. Students 11 years old know this date from history lessons. They are asked to analyse carefully the whole picture (e.g. in the centre they can recognise the king!) and to answer a number of questions: Is it possible that the king Vladislav Jagiello was present that day on the Market Square in Cracov? Is it possible that one of three students has his birthday that day? Find in the picture some examples of things that could be impossible, possible or certain at that time.

FIGURE 1

This example refers to the situation which is embedded in a historical context. Its consideration leads to posing various hypotheses of a probabilistic nature. The main aim of this activity is to enable students to distinguish and to develop the *epistemological* aspect of probability, necessary to form its dual nature, and also to make a qualitative evaluation of a "degree of certainty". Children formulate their opinions, discuss and convince each other. (LAKOMA 1999)

In order to express a "degree of possibility" they use the "measure of chances" – the model of interval (Figure 2). They decide where on this interval "clouds" containing various events could be hung and they try to justify their decisions :

A: I can see things, which were not known that time – air plane, radio, ice cream etc.

B: "3" – small chances, they [the vendors] are rather of various ages. "2"–equal chances, the horse could be black or white. "1"–large chances.

A: "1"– equal chances – rain was also possible; one man has an umbrella!

B: Did an umbrella exist that time? We cannot know for sure. But in summer there is the advantage of sun.

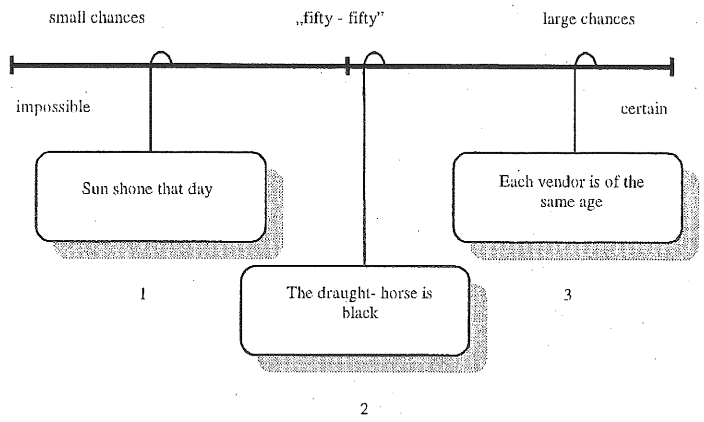

FIGURE 2

Children use different mental objects connected with probability. They describe the chances of events on the base of their belief concerning the frequency of events (in summer *advantage of sun*) – aleatory aspect of probability. They also use the concept of symmetry (*equal chances – black or white*) – epistemological aspect of probability. Exposing arguments *de dicto* sometimes leads to distinguishing only two possible outcomes and to describing their chances just as even: *rain or no rain, horse – black or white* etc. This is a good point of departure to find more information concerning the phenomenon under consideration. Often it is also possible to introduce arguments *de re*. For example, in a case of a draught-horse, students try to estimate the chances of meeting a black horse – by considering a population of horses and estimating the percentage of different colours. They usually evoke their own experience or knowledge concerning colours of horses; sometimes they feel a need to find more information in lexicon. During such a discussion students use both kinds of arguments: *de dicto* and *de re*, although the nature of this example is inclined to develop an *epistemological* aspect of probability. They usually realise that their conclusions are not sure, are only probable, so they describe events as *certain* rather rarely. It is much easier for them to call some events *impossible*. Making the decision where to place a "cloud" representing an event on a measure of chances is rather difficult for pupils; sometimes they do it rather arbitrarily. But some pupils try to find a reasonable explanation – mainly in the categories : *large chances* or *small chances* – and to indicate a place on "the measure of chances". Of course these reasonings are very simple and naive but this is a good opportunity to gain the experience necessary to form in a further stages of education the mature probabilistic concepts in a proper way.

B. "The measure of chances" – two dice

The main aim of this activity is to encourage students to estimate the chances of some events connected with a throw of two dice. Students 13 years old are asked to place on "the measure of chances" (Figure 2) the following "clouds" indicating events:

1) The number 24 will appear
2) The number 78 will appear
3) A number not greater than 67 will appear
4) A number not less than 10 will appear
5) A number, whose digits are equal, will appear
6) A number, whose difference between digits is equal to 1, will appear
7) A number divisible by 3 will appear
8) A number divisible by 5 will appear

Usually pupils consider this example in different steps. First they make only some qualitative estimations of chances, which are based just on observation of throws two dice. Usually, however, they try to support their predictions *de re* by means of considering step-by-step possible outcomes of a throw of two dice (arguments *de dicto*). They use implicitly a model of even chances, based on symmetry of a random mechanism. In these reasonings a need to make some quantitative estimation and to identify a chance with some number strongly appears. Mainly, students start to obtain such numbers by specifying and counting all possibilities promoting the given event. For example : event 8) is promoted by numbers : 15, 25, 35, 45, 55, 65. So, there are 6 *possibilities*. Then, event 1) is represented only by *1 possibility*. Thus, event 1) has smaller chances than event 8). On "the measure of chances" it will be placed much more on the left side than event 8). This comparison relays on students' conviction that *all possibilities are equal* – a die is *symmetric*.

In many students' reasonings we can also observe that in order to estimate chances of an event they choose a number belonging to the interval $< 0; 1 >$. It is rather typical when students are able to compute how many possibilities of outcomes represent their result – in comparison to the whole number of possibilities. So, they create mental objects *such as absolute theoretical frequencies* or *relative theoretical frequencies*. However, it is important to stress that all these estimations are strongly confronted with *experimental frequencies* obtained during introductory observations of throws of two dice. All these arguments support each other.

C. Game "Crossing the River"

You need: 2 × 10 counters, 2 × 1 gameboard (Figure 3), 3 circle chips of different colours – on each chip number "1" is written on one side and number "2" on the other.

This is a game for two persons or two teams. Each of two players -"leaders"- obtains the gameboard (Figure 4) and 10 counters -"competitors". At the beginning of the game each player puts his/her "competitors" on places 1-8 of the gameboard. It is allowed to put more than one "competitor" on a chosen place. Afterwards three "1-2" chips are thrown.

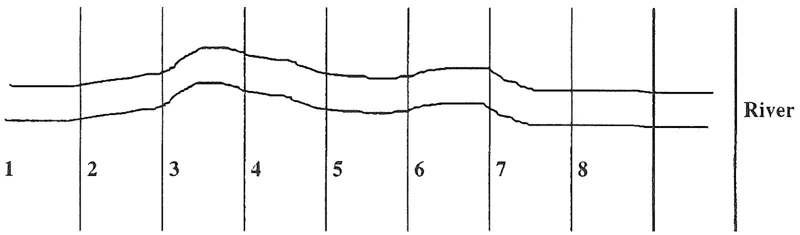

FIGURE 3

The sum of numbers obtained (e.g. 2+2+1) points to a place number (i.e. 5) on the gameboard. If player has any "competitor" standing on this place, he/she moves it (only one "competitor"!) to the other bank of the River. This player who first moves all 10 "competitors" to the other bank of the River wins.

The natural problem which arises in connection with this game is as follows : How to distribute "competitors" in the best way? What places are worth choosing?

Pupils 11 years old decided to create two teams and to play on the school terrace. Their preliminary distributions are :

team A : "3" – 2, "4" – 2, "5" – 2, "6" – 2[1]   team B : "3" – 1, "4" – 3, "5" – 2, "6" – 2.

Team A won this game. In the next game, team B exactly followed the distributions proposed by team A. In opinion of team B this is a warrant of a fair game. [2] Searching for symmetry is typical at this stage of considering this game. There is symmetry of distribution of players within a team, and also symmetry in distribution of whole teams. Searching for symmetry helps students to understand the rules of this game and to work out a strategy to win (LAKOMA 1998).

The pupils work in pairs with paper and pencil. They observe that "4"s are the most frequent. Some of the pupils place all "competitors" on this position. [3]

Pupils draw on the blackboard a bar diagram of all results: "3"-26, "4"-38, "5"-56, "6"-17.

Simultaneously they comment upon these results:
Gabi: "5"s are leaders. [4]
Reni: But "4" has only one point disadvantage in comparison with "5". [5]
Jacob: Now "3" and "5" are the best positions. [6]
Gabi: But there is only one point of difference between them. [7]

When the diagram is finished, pupils try to give a good advice to a player :
Gabi: Place on "3" at most five "competitors", maybe less; on "4" about six or seven; and on "5" ten, maybe more. [9]
Thomas: I propose to place most competitors on "5". [10]

The first pupils' attempts to predict results of the game have an *epistemological* character and are based on the simple symmetry of the situation. But it turns out soon that both models of even chances [1, 2] are not adequate and pupils decide to make conclusions based only on experimental frequencies of results - their reasonings have an *aleatory* character [3, 9, 10]. Pupils realise that gathering data should be reliable [8, 5, 7], they feel a need of comparisons during analysing data [4-7, 9-10], they discover that predictions based on experimental data can't be categorical [5, 7, 10]. Though children use mental objects, which expose both aspects of probability separately, they have good conditions for developing the probability concept according to its dual nature.

Students 12 years old arranged the 2-teams-game in a classroom. During the game they remark that "4" appears often and "6" rarely. [1] Students say that it is difficult to obtain "6", but easy to obtain "4". [2] So they formulate a hint for players: to place "competitors" on "4". [3] Then students work in pairs. They notice that results "1", "2" and "7", "8" are impossible. [4] They distribute "competitors" on places "3" - "6" doing it just by chance or according to the previous observations. [5]

Students write on the blackboard the preliminary distributions of winners :

"3"-3, "4"-4, "5"-2, "6"-1;   "3"-2, "4"-5, "5"-3, "6"-0;   "3"-3, "4"-3, "5"-4, "6"-0 [6].

Maggy: I know that I shouldn't place ["competitors"] : on 1,2,7,8 because these numbers never appear. The best way to distribute "competitors" doesn't exist.
This is just luck. [7]
Mary: But there are numbers, which appear very rarely. So it is good to place few "competitors" for "3", most of all on "4" and "5", and a few on "6". [8]
Thom: No, no! To pose a few on "3" but nothing on "6"! [9]
Peter: But what is the difference between "3" and "6"? There is no difference! [10]
Mary: I would like to pose most "competitors" on "4" and "5" because it is better chance to obtain 2-2-1 than 2-2-2. [11]
Peter: or 1-1-1! [12]

Students notice that there are more mixed results than pure ones.
Thom: Thus we would place most counters on "4" and "5",
hardly any on "6", a few on "3". But this is a query of luck.

These reasonings are more advanced than those done by pupils 11 years old. On the ground of empirical frequencies students notice that some results are "impossible", "possible", "difficult" or "easy" to obtain [1-4]. These are reasonings in style of Cardano and Galileo (see LAKOMA, 1992). Students try to make conscious the symmetry of the situation [4, 7] but they don't use it during the game [4, 6, 7]. During discussion students already notice this symmetry of situation [8,10, 12].

It is evident that these interactions among students help them to develop their individual ways of thinking [8]. The influence of Peter on Thom's reasoning is very spectacular. Also Mary tries to use not only *aleatory* but also some *epistemological* arguments. Both these aspects reinforce each other in the final part of the discussion [8-13]. It seems that for these students it can be only one step to find an adequate model and to calculate chances or theoretical frequencies of all results. It seems also that nobody in this group is able to solve this problem individually - without any co-operation with other students.

Let us observe the solution of two students 15 years old : Martin and Christopher. Their preliminary distributions are :
Ch: "3"- 2, "4" - 4, "5" - 2, "6" - 2 M: "3"-2, "4"-3, "5"-3, "6"-2 [1]
They motivate their choices to each other :
Ch: I already know what results [sums of numbers obtained on 3 chips] I should avoid : "1" and "2" because I will never obtain them. [2]
M: It is possible to obtain : "3", "4", "5" or "6" but "7" and "8" are also impossible. [3]

They play the game recording all results obtained. They are very engaged in playing.

Chris has lost this game. Both players try to find the reason for it. They consider all results obtained :
Ch: "3"-2, "4"-6, "5"-4, "6"-2 M: "3"-3, "4"-3, "5"-5, "6"-3
Ch: I think I should place more "counters" on "3". [4]
M: I obtained different results from yours! But "3"s and "6"s are also the rarest. [5]
Ch: This might be because you had crooked chips. Look, one should avoid "3" and "6" because these results are rare. [6]
M: Yes, maybe. But, you know, I obtained "3" three times! [7]
Ch: Maybe it happened by chance - you can check it in other games. "5"s and "4"s - there are many of them. And "3"s and "6"s should be avoided. [8]

M: It is necessary to put most chips on the two middle positions. [9]

Chris is convinced that his results are "typical" for this game. He classifies Martin's results as extraordinary. Martin eventually agrees with Chris. They both return to the question : how to distribute the counters in an optimal way?
Ch: I pose one counter on "3", and one on "6" - because they happen rarely. Wait, maybe I will throw "6"... but it will not happen; there is no chance ... number 2 on every chip?? [10]
M: I place ["competitors"] on "4" and "5". We have mostly "4" and "5". [11]
Ch: Yes, so I propose; one counter on "3", four on "4", four on "5" and one on "6". [12]

They formulate their opinion :
M: The best way is to place most counters on the middle positions, because the extreme positions are the rarest. [13]
Ch: "6" is the rare position because it happens very rarely - when it is obtained only number 1 or only number 2 on every chip. [14]
M: We can distribute our "competitors" in parts : on place "3" 1/10 of all "competitors", on "4" 4/10, on "5" 4/10, and on "6" 1/10. That means - not always so exactly. We place the fewest counters on the extreme places and most on the middle places. It arises from our experience concerning this game. Mixed results are the most frequent. [15]

After some hours it turns out that Martin during his next look on the problem has doubts :
M: Now I think that counters should be distributed in equal parts because there is no power which could say that it is necessary to obtain number 1 or number 2. [16]

The first attempts to consider the game were based on epistemological reasonings: searching for any symmetry [1], eliminating impossible results [2,3]. Then the *aleatory* aspect dominates in students' arguments [4-9]. Nevertheless students try to find an *epistemological* basis for their opinions. They search for symmetry again [6,9], they distinguish "typical" and "extraordinary" results - their motivations at this stage contain already both aspects of probability [10, 12, 14]. Martin tries to find an adequate model. He has good intuitions - expressing distribution by means of fractions is a good way although not yet proper [15, 16]. His reasoning *in style of d'Alembert* [16] (LAKOMA, 1992) confirms that he feels a need to consider both the nature of random phenomenon and its theoretical model.

D. Shooting at the target :

A shooter hits the target with frequency $\frac{1}{2}$. He shoots until he hits this target.

How many trials – most probably – should he make in order to hit?

If we like, we can formulate this problem in another way and arrange any real everyday life situation in which students repeat some trials (which are successful or not with even odds) in order to gain some aim. When (in which trial) they have the best possibility to obtain what they need?

It is very important to ask students to express –just after formulating the problem– their predictions. The most common answer is as follows: three, four, or two.

Then, in order to get acquainted with this problem, students make a Monte Carlo simulation, for example by repetitions of randomly drawing a ball (white or blue) from a basket. (When you draw a blue ball, you have gained your aim, if not, you put the white ball back into the basket and you repeat your trial again). They observe how many trials they needed to obtain a blue ball.

After some experiments they realise that the most common number of trials to obtain a blue ball is just one. They are confused. Is it really true?

Their arguments *de dicto* contradict the results of their experiment (*de re*). In order to check if their observations are typical, students try to construct a local model of the phenomenon and compute on this basis the chances of gaining an aim after one, or two etc. trials. Drawing a tree is already available for pupils 13 years old (Figure 4). This model helps them to understand the reason of apparent contradiction of both aspects. After considering this tree it is confirmed that the most probable trial is the first one. But this answer appears very rarely when we ask pupils, students, mathematicians or even probability theory experts – before an experiment. Students treat the right answer as a paradox. The reason for this wrong intuition is that they use the concept of *expected value* rather than the concept of *probability*. But developing both aspects of probabilistic reasoning helps to overcome this obstacle. It is possible to ask older students to compute an expected value *EX* of the number of trials needed to gain an aim (Figure 4).

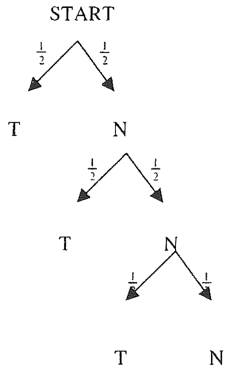

FIGURE 4

$$EX = \sum x_i \cdot p_i$$

$$\begin{array}{rcl}
1\cdot\frac{1}{2}+2\cdot\frac{1}{4}+3\cdot\frac{1}{8}+4\cdot\frac{1}{16}+\ldots & = & EX \\
\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots & = & 1 \\
\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots & = & \frac{1}{2} \\
\frac{1}{8}+\frac{1}{16}+\cdots & = & \frac{1}{4} \\
\frac{1}{16}+\cdots & = & \frac{1}{8} \\
 & & \vdots \\
 & & \downarrow \\
 & & 2
\end{array}$$

FIGURE 5

Final remarks

All the examples which we had the opportunity to analyse above show that the process of forming the dual probability concept is according to the cognitive development of students. Natural ways of a student's reasoning –followed in these examples– convince us that the process of development of probabilistic thinking is accompanied in a natural way by using both aspects of probabilistic arguments – *de re* and *de dicto*. It is strongly connected with the natural need to observe the phenomenon under consideration and to make efforts to build its model. These activities allow students to understand the essence of *mathematical modelling*. This competence lets them analyse real phenomena or regularities by means of *mathematical reasoning*, making conclusions on the basis of a model, posing hypotheses and verifying them. What is important is an ability to explain and a practical efficiency (DAVIS & HERSH 1981, LAKATOS 1976). Creating and analysing models, accompanied by empirical observations, mathematical thinking and mathematical reasoning must be expressed and explained in a clear way. This competence –*mathematical communication*– is the third fundamental aspect of mathematics that must be developed, in a necessary balance, during the process of mathematics learning. Learning probability seems to expose these aspects in a spectacular way.

It seems that in order to encourage students to learn probability according to their own individual cognitive development it is very useful to create didactical situations for which various forms of *interactions* among students will be the natural style of work (SIERPINSKA & KILPATRICK 1998, LAKOMA 1998, BROEKMAN 1995). The main aim of provoking interactions in a classroom is to stimulate students to think independently, to formulate their own opinions, to motivate these opinions and communicate them to other students, to make conscious errors, to discover their own natural ways of thinking, to create and use a natural language for communicating with each other (FREUDENTHAL 1983, SIERPINSKA 1994). All these activities seem to be especially fruitful because they contribute to develop pupils' probabilistic thinking and to create in their minds the concept of probability in its mature dual form. All presented forms of interactions among student -working in teams, in pairs, competing in the game, preparing a common solution of a problem, discussions in a small group or in a whole class- seem necessary to the process of probability learning at every level of education. They evidently influence students' individual development of probabilistic thinking.

From the examples shown above let us also notice that interactions among students encourage them to be more effective in the process of problem solving. Using parallel representations of a problem, explaining this problem from various points of view, increase student's progress in developing probability concepts.

The duality of probability concept, which seems to be evident in students' activities, plays a very important role when we consider the process of probability learning from the point of view of a teacher. Using both aspects of probability in student's arguments, supporting and reinforcing each other seems to mean that a student is able to create and to use various mental objects which in the future will coalesce and will form the mature probability concept. Thus, a teacher –thanks to this essential feature of probability– is able to recognise the student's way of reasoning, to evaluate it and to organise such mathematical activities, as can help him to make his learning more effective.

The ideas which I have briefly presented here have been studied for some years in the frame of an educational project for students of age 10-16 (LAKOMA, 1990, LAKOMA & ZAWADOWSKI a.o., 1996-2000) and, at experimental stage of research, at tertiary level, for future professional users of mathematics : teachers, engineers and economists (LAKOMA, 1998a).

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